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On Systems of Renewal Equations: The Reducible Case

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1. INTRODUCTION

Consider the system of renewal-type integral equations

$$M_i(t) = z_i(t) + \int_0^t \sum_{j=1}^m M_j(t-u) F_{ij}(du), \qquad i = 1, 2, \ldots, m, \tag{1.1}$$

where each $z_i(t)$ is a given measurable function bounded on every finite interval and vanishing for $t < 0$ and each $F_{ij}(t)$ is a distribution function that vanishes for $t < 0$. If we let $\mathscr{F}(t)$ be the $m \times m$ matrix $(F_{ij}(t))$, $\mathscr{Z}(t)$ the column vector $(z_1(t),\ldots, z_m(t))'$ and $\mathscr{M}(t)$ the column vector $(M_1(t),\ldots, M_m(t))'$, equation (1.1) may be written as

$$\mathscr{M}(t) = \mathscr{Z}(t) + \mathscr{F} * \mathscr{M}(t), \tag{1.2}$$

where the operation "$*$" of convolution of two matrices is defined in the same way as matrix multiplication except that we convolve elements rather than multiply them.

The asymptotic behavior of the solution vector $\mathscr{M}(t)$ has been determined in [2] in certain cases of interest under the assumption the matrix $\mathscr{F}(\infty)$ is irreducible (the irreducible case). More specifically, it was established in [2] that under certain regularity conditions $M_i(t) \sim C_i e^{\alpha t}$ as $t \to \infty$ where α is a constant that does not depend on i. The purpose of the present paper is to extend the results in [2] to allow $\mathscr{F}(\infty)$ to be a reducible matrix (the reducible case). As we shall see, the reducible case yields a much richer theory. By dropping only the assumption that $\mathscr{F}(\infty)$ is irreducible we shall obtain limiting expressions of the type $M_i(t) \sim C_i t^{r_i} e^{\alpha_i t}$ where C_i, r_i, and α_i may all depend on i.

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The system of integral equations (1.1) has applications in various types of age-dependent branching processes. In Section 4 we will deduce some limiting expressions for the first moments of a reducible (decomposable) multidimensional age-dependent branching process.

2. The Irreducible Case

In this section we shall present some results for the irreducible case, some of which are new, which we will need for the investigation of the reducible case.

A square matrix $\mathscr{A}$ with nonnegative entries is said to be reducible if there is a permutation of the columns of $\mathscr{A}$ that, together with the same permutation of the rows of $\mathscr{A}$, puts $\mathscr{A}$ into the form

$$\hat{\mathscr{A}} = \begin{pmatrix} \mathscr{B} & \mathscr{O} \\ \mathscr{C} & \mathscr{D} \end{pmatrix},$$

where $\mathscr{B}$ and $\mathscr{D}$ are square matrices. Otherwise $\mathscr{A}$ is said to be irreducible. It should be pointed out that although it was assumed in [2] that each entry in the matrix $\mathscr{F}(\infty)$ was strictly positive, all of the arguments go over without change under the weaker assumptions that $\mathscr{F}(\infty)$ is irreducible and either at least one of the distribution functions $F_{ii}(t)$ on the main diagonal has a point of increase or at least one of the off-diagonal distribution functions $F_{ij}(t)$, $i \neq j$, has at least two points of increase.

Lemma 2.1. *Suppose $z(t)$ is bounded on finite intervals, vanishes for $t < 0$, and $z(t)/t^k \to c$ as $t \to \infty$ for some nonnegative integer k and finite constant c. Suppose also that $F(t)$ is a distribution function that vanishes for $t < 0$.*

(i) *If $F(\infty) < \infty$, then*

$$\frac{1}{t^k} \int_0^t z(t-u) F(du) \to cF(\infty) \qquad as \qquad t \to \infty.$$

(ii) *If $F(t)/t \to b$ as $t \to \infty$ then*

$$\frac{1}{t^{k+1}} \int_0^t z(t-u) F(du) \to \frac{bc}{k+1} \qquad as \qquad t \to \infty.$$

Proof. Both (i) and (ii) are elementary and are probably well-known. However, it turned out that the author could not find the proof of either in the literature. Since the proof of (ii) is slightly harder we shall give it below and omit the proof of (i).

By breaking the integral up into the sum of the integrals from 0 to $t - t'$ and from $t - t'$ to t and by adding and subtracting

$$\frac{bc}{t^{k+1}} \int_0^{t-t'} (t-u)^k \, du$$

we obtain

$$\left| \frac{1}{t^{k+1}} \int_0^t z(t-u) F(du) - \frac{bc}{k+1} \right| \leqslant \frac{1}{t} \int_0^{t-t'} \left| \frac{z(t-u)}{(t-u)^k} - c \right| \left(\frac{t-u}{t} \right)^k F(du)$$
$$+ \frac{|c|}{t^{k+1}} \left| \int_0^{t-t'} (t-u)^k F(du) \right.$$
$$\left. - b \int_0^{t-t'} (t-u)^k \, du \right| \qquad (2.1)$$
$$+ \frac{|bc|}{t^{k+1}} \left| \int_0^{t-t'} (t-u)^k \, du - \frac{1}{k+1} \right|$$
$$+ \frac{1}{t^{k+1}} \int_{t-t'}^t |z(t-u)| \, F(du).$$

Choose t' so large that $|z(t)/t^k - c| < \epsilon$ for $t \geqslant t'$. Then for $t > t'$ the first term on the right side of (2.1) is

$$\leqslant \frac{\epsilon}{t} \int_0^{t-t'} \left(\frac{t-u}{t} \right)^k F(du) \leqslant \epsilon \frac{F(t)}{t}.$$

One sees by direct integration that the third term on the right side of (2.1) approaches zero as $t \to \infty$. The last term is

$$\leqslant \frac{1}{t} \int_{t-t'}^t \left| \frac{z(t-u)}{(t-u)^k} \right| \left(\frac{t-u}{t} \right)^k F(du)$$
$$\leqslant \sup_{0 \leqslant x < \infty} \left(\frac{z(x)}{x^k} \right) \frac{1}{t} (F(t) - F(t-t')) \to 0 \qquad \text{as} \qquad t \to \infty.$$

By integrating by parts we get the second term on the right side of (2.1) to be

$$\leqslant \frac{|c|(t')^k}{t^{k+1}} |F(t-t') - b(t-t')| + \frac{k|c|}{t^{k+1}} \int_0^{t-t'} |F(u) - bu| (t-u)^{k-1} \, du.$$

The first term in this expression clearly approaches zero as $t \to \infty$ and for $k \geqslant 1$ the second term is

$$\leqslant \frac{k|c|}{t^2} [F(t) + bt) \to 0 \qquad \text{as} \qquad t \to \infty.$$

Similar arguments apply when $k = 0$.

We shall also need the following notion used by Feller [4] in proving the key renewal theorem in a very general form. For a fixed $h > 0$, let m_n and M_n denote the minimum and maximum, respectively, in the interval $(n-1)\,h \leqslant t \leqslant nh$ of a function $z(t)$ that vanishes for $t < 0$, and let $s = h\Sigma m_n$ and $S = h\Sigma M_n$. Feller calls $z(t)$ "directly Riemann integrable" whenever these sums converge absolutely and $S - s < \epsilon$ for h sufficiently small.

The following two lemmas will be needed in the proof of Theorem 3.1.

LEMMA 2.2. *If $z(t)$, $t \geqslant 0$, is a nonnegative function which is Riemann integrable and is dominated by a monotone function which is integrable on* $[0, \infty)$ *then $z(t)$ is directly Riemann integrable.*

Proof. The proof is elementary and will be omitted.

LEMMA 2.3. *If $z(t)$ is Riemann integrable on finite intervals and the distribution function $F(t)$ has no singular component, then $f(t) = \int z(t-u)\,F(du)$ is Riemann integrable on finite intervals.*

Proof. We shall use the fact that a function is Riemann integrable if and only if its set of discontinuities has Lebesque measure zero.

Using Fatou's lemma we see that

$$L(t) = \limsup_{h\to 0}\, |f(t+h) - f(t)| \leqslant \int s(t-u)\,F(du),$$

where

$$s(t) = \limsup_{h\to 0}\, |\, z(t+h) - z(t)|\,.$$

By decomposing F into its absolutely continuous part F_{uc} and its atomic part F_d we get

$$L(t) \leqslant \int_{D_t} s(t-u)\,F_d(du),$$

where D_t is the u set on which $s(t-u)$ is positive. Therefore, $L(t) = 0$ unless $t = a + b$ where a is a point for which $s(a)$ is positive and b is a point of discontinuity of $F_d(t)$. However $F_d(t)$ can have only a countable number of discontinuities and so the set of such t's has Lebesque measure zero.

When $\mathscr{F}(\infty)$ is an irreducible matrix limit theorems for the vector $\mathscr{M}(t)$ must be given in two parts depending on whether or not, using the terminology of [2], $\mathscr{F}(t)$ is a "lattice matrix." Although the same dichotomy may be made when $\mathscr{F}(\infty)$ is reducible, in order to simplify matters we shall consider only the nonlattice case since it seems to be by far the most important. The reader may consult [2] for the definition of a lattice matrix. We shall only note that if at least one distribution functions $F_{ij}(t)$ has an absolutely continuous part then the matrix $\mathscr{F}(t)$ is not a lattice matrix.

In the remainder of the paper we shall assume therefore that $\mathscr{F}(t)$ is not a lattice matrix. As in [2] we shall also assume that for some i and j the distribution function $F_{ij}(t)$ is not concentrated at the origin and that $\rho(\mathscr{F}(0)) < 1$ where $\rho(\mathscr{A})$ denotes the largest eigenvalue of the matrix $\mathscr{A}$.

Define recursively the matrices $\mathscr{F}^{(n)}(t)$ by $\mathscr{F}^{(0)}(t) = (\delta_{ij}(t))$ where $\delta_{ij}(t) = 1$ if $t \geqslant 0$ and $i = j$ and $\delta_{ij}(t) = 0$ otherwise, $\mathscr{F}^{(1)}(t) = \mathscr{F}(t)$ and $\mathscr{F}^{(n+1)}(t) = \mathscr{F}^{(n)} * \mathscr{F}(t)$. Now put $\mathscr{U}(t) = \sum_{n=0}^{\infty} \mathscr{F}^{(n)}(t)$. Furthermore let $G_{ij}(s) = \int_0^{\infty} e^{-su} F_{ij}(du)$, $\mathscr{G}(s) = (G_{ij}(s))$, denote the determinant of $\mathscr{I} - \mathscr{G}(s)$ by $\Delta(s)$ and let $(c_{ij}(s))$ be the adjoint matrix of $\mathscr{I} - \mathscr{G}(s)$ where $\mathscr{I}$ is an $m \times m$ identity matrix. Define α to be the unique number (if it exists) that makes $\rho(\mathscr{G}(\alpha)) = 1$. Observe that α always exists and is positive if $\mathscr{F}(\infty)$ has finite entries and $\rho(\mathscr{F}(\infty)) > 1$, $\alpha = 0$ if $\rho(\mathscr{F}(\infty)) = 1$, and $\alpha < 0$, provided it exists, for $\rho(\mathscr{F}(\infty)) < 1$. Finally, let $\mathscr{B}(\alpha)$ be the matrix whose ij-th entry is

$$b_{ij}(\alpha) = c_{ij}(s) \Big/ \frac{d}{ds} \Delta(s) \Big|_{s=\alpha}$$

whenever $\int_0^{\infty} te^{-\alpha t} F_{ij}(dt) < \infty$ for each i and j. If this integral is infinite for some i and j let $\mathscr{B}(\alpha)$ be the zero matrix.

We shall also use the following matrix notations. If $\mathscr{A}(t) = (A_{ij}(t))$ and $\mathscr{A} = (A_{ij})$ then $\mathscr{A}(t) \to \mathscr{A}$ as $t \to \infty$ means, as usual, that $A_{ij}(t) \to A_{ij}$ as $t \to \infty$ for each i and j. The symbol $\int \mathscr{A}(t)$ will denote the matrix whose ij-th entry is the Lebesque integral on $[0, \infty)$ of the ij-th entry of $\mathscr{A}(t)$.

THEOREM 2.1.

(i) *The integral equation* (1.2) *has a unique solution that is bounded on finite intervals and vanishes for* $t < 0$. *This solution is*

$$\mathscr{M}(t) = \mathscr{U} * \mathscr{Z}(t).$$

(ii) *If* $\mathscr{F}(\infty)$ *is irreducible and* $\rho(\mathscr{F}(\infty)) = 1$, *then*

$$\mathscr{U}(t + h) - \mathscr{U}(t) \to h\mathscr{B}(0) \qquad as \qquad t \to \infty$$

for each positive h.

(iii) *If* $\mathscr{F}(\infty)$ *is irreducible, the number* α *exists, and* $e^{-\alpha t} z_i(t)$ *is directly Riemann integrable for each* i *then*

$$e^{-\alpha t}\mathscr{M}(t) \to \mathscr{B}(\alpha) \int e^{-\alpha u} \mathscr{Z}(u) \qquad as \qquad t \to \infty.$$

(iv) *If* $\mathscr{F}(\infty)$ *is irreducible, the number* α *exists, and*

$$\frac{1}{t^k e^{at}}\, \mathscr{Z}(t) \to \mathscr{C} \qquad as \qquad t \to \infty$$

for some vector $\mathscr{C}$, *nonnegative integer* k, *and number* $a > \alpha$, *then*

$$\frac{1}{t^k e^{at}}\, \mathscr{M}(t) \to [\mathscr{I} - \mathscr{G}(a)]^{-1}\, \mathscr{C} \qquad as \qquad t \to \infty.$$

(v) *Under the same hypotheses as* (iv) *except that* $\alpha = a$, *we have*

$$\frac{1}{t^{k+1} e^{\alpha t}}\, \mathscr{M}(t) \to \frac{1}{k+1}\, \mathscr{B}(\alpha)\, \mathscr{C} \qquad as \qquad t \to \infty.$$

Proof. Parts (i), (ii) and (iii) are restatements of facts from [2]. To prove (iv) let $\hat{\mathscr{Z}}(t) = e^{-at}\mathscr{Z}(t)$, $\hat{\mathscr{M}}(t) = e^{-at}\mathscr{M}(t)$ and let $\hat{\mathscr{F}}(t)$ be the $m \times m$ matrix whose ij-th component is

$$\hat{F}_{ij}(t) = \int_0^t e^{-au} F_{ij}(du).$$

Then it easy to show that $\hat{\mathscr{M}}(t)$ satisfies

$$\hat{\mathscr{M}}(t) = \hat{\mathscr{Z}}(t) + \hat{\mathscr{F}} * \hat{\mathscr{M}}(t),$$

and that $\hat{\mathscr{F}}^{(n)}(t)$, the n-fold convolution of the matrix $\hat{\mathscr{F}}(t)$ with itself, has $\int_0^t e^{-au} F_{ij}^{(n)}(du)$ as its ij-th element. Therefore, by (i)

$$\hat{\mathscr{M}}(t) = \hat{\mathscr{U}} * \hat{\mathscr{Z}}(t) \tag{2.2}$$

where

$$\hat{\mathscr{U}}(t) = \sum_{n=0}^{\infty} \hat{\mathscr{F}}(t).$$

Since

$$\rho(\hat{\mathscr{F}}(\infty)) < 1, \qquad \hat{\mathscr{U}}(\infty) = \sum_{n=0}^{\infty} \hat{\mathscr{F}}(\infty) = [\mathscr{I} - \mathscr{G}(a)]^{-1}.$$

If we now apply part (i) of Lemma 2.1 to the i-th component of the vector equation (2.2) we get

$$\frac{M_i(t)}{t^k e^{at}} = \sum_{j=1}^{m} \frac{1}{t^k} \int_0^t \hat{z}_j(t-u)\, \hat{U}_{ij}(du) \to \sum_{j=1}^{m} \hat{U}_{ij}(\infty)\, c_j \qquad \text{as} \qquad t \to \infty, \tag{2.3}$$

where $\hat{z}_i(t)$, $\hat{U}_{ij}(t)$, and c_j denote, respectively, the components of $\hat{\mathscr{Z}}(t)$, $\hat{\mathscr{U}}(t)$, and $\mathscr{C}$.

To prove (v) we proceed in exactly the same way as we did in the proof of (iv) except we replace a by α, replace k by $k+1$ in (2.3), appeal to part (ii) of Lemma 2.1 rather than part (i) and use the fact that by part (i) of Theorem 2.1.

$$\frac{1}{t}\hat{\mathscr{U}}(t) \to \mathscr{B}(\alpha) \qquad \text{as} \qquad t \to \infty.$$

3. The Reducible Case

If $\mathscr{F}(\infty)$ is reducible, then (Gantmacher [5]) by a permutation of the rows of $\mathscr{F}(t)$ together with the same permutation of its columns, $\mathscr{F}(\infty)$ may be put into the form,

$$\mathscr{F}(\infty) = \begin{pmatrix} \mathscr{F}_{11}(\infty) & 0 & \cdots & 0 \\ \mathscr{F}_{21}(\infty) & \mathscr{F}_{22}(\infty) & \cdots & 0 \\ \vdots & & & \\ \mathscr{F}_{d1}(\infty) & \mathscr{F}_{d2}(\infty) & \cdots & \mathscr{F}_{dd}(\infty) \end{pmatrix}, \tag{3.1}$$

where each $\mathscr{F}_{ii}(\infty)$ is either a 1×1 zero matrix or is a square irreducible matrix. Note that if we apply the same permutation to the components of the vectors $\mathscr{M}(t)$ and $\mathscr{Z}(t)$ then the system of integral equations (1.1) is unchanged. We shall assume from now on that these permutations have been completed and also that no matrix $\mathscr{F}_{ii}(\infty)$ on the main diagonal is a zero matrix.

The representation (3.1) induces a partition of the integers $1, 2,..., m$ into d disjoint classes $C_1 ,..., C_d$ so that

$$\mathscr{F}_{ab}(\infty) = (F_{ij}(\infty))_{i \in C_a, j \in C_b} .$$

For $a, b = 1, 2,..., d$ we define

$$\mathscr{M}_a(t) = (M_i(t))_{i \in C_a}, \qquad \mathscr{Z}_a(t) = (Z_i(t))_{i \in C_a}, \qquad \mathscr{F}_{ab}(t) = (F_{ij}(t))_{i \in C_a, j \in C_b},$$

$$\mathscr{G}_{ab}(s) = (G_{ij}(s))_{i \in C_a, j \in C_b}$$

and $\mathscr{B}_a(\alpha)$ to be the matrix derived from $\mathscr{F}_{aa}(t)$ in the same way that $\mathscr{B}(\alpha)$ is derived from $\mathscr{F}(t)$. We shall say that "a follows b" (in symbols, $b \to a$) if for some j there are submatrices $\mathscr{F}_{i_1 b}(\infty)$, $\mathscr{F}_{i_2 i_1}(\infty),..., \mathscr{F}_{a i_j}(\infty)$, none of which are zero matrices. We shall say, also, that "a follows b directly" (in symbols $b \Rightarrow a$) if $a \neq b$ and the matrix $\mathscr{F}_{ab}(\infty)$ is not a zero matrix. The system of equations (1.2) can now be written in the form,

$$\mathscr{M}_a(t) = \mathscr{Z}_a(t) + \sum_{b \Rightarrow a} \mathscr{F}_{ab} * \mathscr{M}_b(t) + \mathscr{F}_{aa} * \mathscr{M}_a(t), \qquad a = 1, 2,..., d. \tag{3.2}$$

Since each $\mathscr{F}_{aa}(\infty)$ is an irreducible matrix we shall be able to apply parts (iii), (iv), and (v) of Theorem 2.1 to (3.2) to determine the asymptotic behaviour of the vector $\mathscr{M}_a(t)$.

Let α_a be the number (if it exists) for which $\rho(\mathscr{G}_{aa}(\alpha_a)) = 1$, let $\beta_a = \max\{\alpha_i : i \rightarrow a\}$ and let r_a be the largest number of distinct indices $i_1 ,..., i_{r_a}$ such that $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_{r_a} \rightarrow a$ and $\beta_a = \alpha_{i_1} = \cdots = \alpha_{i_{r_a}}$. Note that r_a may equal but cannot exceed a.

THEOREM 3.1. *Suppose the number α_a exists for each a, each matrix $\mathscr{G}_{ab}(\beta_a)$, $a \neq b$ has all finite entries, the components of the vector $e^{-\alpha_a t}\mathscr{Z}_a(t)$ are directly Riemann integrable for each a, and none of the distribution functions $F_{ij}(t)$, $i, j = 1, 2,..., m$ have a singular component. Then*

(i) *if $r_a = 1$ (i.e., $\alpha_a > \alpha_b$ for all b for which $b \Rightarrow a$) then $\alpha_a = \beta_a$ and*

$$e^{-\beta_a t}\mathscr{M}_a(t) \rightarrow \mathscr{B}_a(\beta_a) \int e^{-\beta_a t} \left[\mathscr{Z}_a(t) + \sum_{b \Rightarrow a} \mathscr{F}_{ab} * \mathscr{M}_b(t)\right] \qquad as \qquad t \rightarrow \infty; \tag{3.3}$$

(ii) *if $r_a > 1$ and $\alpha_a < \beta_a$ then*

$$\frac{e^{-\beta_a t}}{t^{r_a - 1}} \mathscr{M}_a(t) \rightarrow [\mathscr{I} - \mathscr{G}_{aa}(\beta_a)]^{-1} \sum_{b \in E_1} \mathscr{G}_{ab}(\beta_a) \lim_{t \to \infty} \frac{e^{-\beta_a t}}{t^{r_a - 1}} \mathscr{M}_b(t) \qquad as \qquad t \rightarrow \infty \tag{3.4}$$

where E_1 is the set of indices b such that $b \Rightarrow a$, $\beta_b = \beta_a$ and $r_b = r_a$;

(iii) *if $r_a > 1$ and $\alpha_a = \beta_a$ then*

$$\frac{e^{-\beta_a t}}{t^{r_a - 1}} \mathscr{M}_a(t) \rightarrow \frac{1}{r_a - 1} \mathscr{B}_a(\beta_a) \sum_{b \in E_2} \mathscr{G}_{ab}(\beta_a) \lim_{t \to \infty} \frac{e^{-\beta_a t}}{t^{r_a - 2}} \mathscr{M}_a(t)$$

where E_2 is defined in the same way as E_1 except r_a is replaced by $r_a - 1$.

Each of the limits in (i), (ii), *and* (iii) *is a vector with finite components. In addition, if for each $a = 1, 2,..., d$ we have $\int te^{-\alpha_a t}F_{ij}(t) < \infty$ for all $i, j \in C_a$, and $\int e^{-\alpha_a t}\mathscr{Z}_a(t)$ is a nonnegative vector for all a with at least one positive component whenever there is no b such that $b \Rightarrow a$, then each of the limits in* (i), (ii), *and* (iii) *is a vector whose components are all positive. Hence, under these conditions the asymptotic form of the vector $\mathscr{M}(t)$ is completely determined.*

Proof. The proof is by induction on a. Assume the theorem is true for all integers less then a (This implies the theorem is true for all b such that $b \Rightarrow a$) and first take the case $r_a = 1$. If there is no b such that $b \Rightarrow a$ (i) follows directly from Theorem 2.1(iii). If however there is such a b then $\beta_b < \beta_a$ and by induction $(e^{-\beta_b t}/t^{r_b - 1}) \mathscr{M}_b(t)$ converges to a finite vector. By applying Lemma 2.1(i) to each of the components we see that $(e^{-\beta_b t}/t^{r_b-1}) \mathscr{F}_{ab} * \mathscr{M}_b(t)$ also converges to a finite vector as $t \to \infty$. By Lemmas 2.2 and 2.3 the components of $e^{-\beta_a t} \mathscr{F}_{ab} * \mathscr{M}_b(t)$ are directly Riemann integrable. The conclusion (i) now follows by applying Theorem 2.1(iii).

Now take case (ii) and suppose $r_a > 1$ and $\alpha_a = \beta_a$. If $b \Rightarrow a$ and $b \notin E_2$ then either $\beta_b < \beta_a$ or $\beta_b = \beta_a$ and $r_b < r_a - 1$. Again by induction $(e^{\beta_b}/t^{r_b - 1}) \mathscr{M}_b(t)$ converges to a finite vector as $t \to \infty$ and by applying Lemma 2.1(i), $[e^{-\beta_a t}/t^{r_a - 2}]\, \mathscr{F}_{ab} * \mathscr{M}_b(t) \to \mathscr{O}$ as $t \to \infty$. Moreover, since the components of $e^{-\beta_a t} \mathscr{Z}_a(t)$ are directly Riemann integrable we know that $(e^{-\beta_a t}/t^{r_a - 1})\, \mathscr{Z}_a(t) \to \mathscr{O}$ as $t \to \infty$. However, if $b \in E_2$ then by induction, using (i) if $r_a = 2$ and (ii) and (iii) if $r_a > 2$, and by Lemma 2.1(i),

$$\frac{e^{-\beta_a t}}{t^{r_a - 2}} \mathscr{F}_{ab} * \mathscr{M}_b(t) \to \mathscr{G}_{ab}(\beta_a) \cdot \mathscr{C}$$

as $t \to \infty$ for some finite vector $\mathscr{C}$. The conclusion (iii) now follows by applying Theorem 2.1(v).

The proof of (ii) follows in a similar fashion by applying (iv) of Theorem 2.1 instead of (v).

To see that the components of the limit vector are all positive in each case note that in (i) and (iii) the matrix $\mathscr{B}_a(\beta_a)$ has all positive entries (see [2]) and that in (ii), since $\mathscr{F}_{aa}(\infty)$ is irreducible, each entry of the matrix $[\mathscr{I} - \mathscr{G}_{aa}(\beta_a)]^{-1}$ is positive, also. Moreover, by induction, the matrices which are premultiplied by $\mathscr{B}_a(\beta_a)$ in (i) and (iii) and $[\mathscr{I} - \mathscr{G}_{aa}(\beta_a)]^{-1}$ in (ii) are all nonnegative with at least one positive component.

4. An Application to a Branching Process

Let us illustrate Theorem 3.1 by applying it to the moments of a reducible (decomposable) age-dependent branching process in which there are m different genotypes. The reader may consult Mode [7] for a more thorough description of the process. Let $m_{ij} < \infty$ denote the expected number of offspring of the j-th genotype of an individual of the i-th genotype, let $M_{ij}(t)$ denote the expected number of individuals of the j-th genotype alive at time t that descend from a single individual of the i-th genotype born at time $t = 0$, and let $G_i(t)$ be the distribution of the life-span of an individual of the i-th genotype. We will assume that each $G_i(t)$ is a proper distribution function which vanishes for $t < 0$ and that at least one $G_i(t)$ is not arithmetic. It may be shown (Mode [7]) that the moments $M_{ij}(t)$ satisfy

$$M_{ij}(t) = \delta_{ij}(1 - G_i(t)) + \sum_{k=1}^{m} \int_0^t m_{ik} M_{kj}(t-u)\, G_i(du), \qquad i, j = 1, 2,..., m. \tag{1.4}$$

Let us suppose that the matrix $\mathscr{A} = (m_{ij})$ is reducible and to keep things simple that

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_{11} & \mathscr{O} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{pmatrix}, \tag{4.2}$$

where $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$ are square irreducible matrices and $\mathscr{A}_{21} \neq \mathscr{O}$. The representation (4.2) implies that the genotypes may be divided into two classes C_1 and C_2 and individuals with genotypes in class C_2 may have offspring whose genotype is in class C_1 but not vice versa. We may write

$$\mathscr{A}_{ab} = (m_{ij})_{i \in C_a, j \in C_b}, \qquad i, j = 1, 2.$$

If we put

$$\mathscr{M}_{aj}(t) = (M_{ij}(t))_{i \in C_a}$$

and

$$\mathscr{Z}_{aj}(t) = (\delta_{ij}(1 - G_j(t)))_{i \in C_a} \qquad \text{for} \qquad a = 1, 2$$

and

$$\mathscr{F}_{ab}(t) = (m_{ij} G_i(t))_{i \in C_a, j \in C_b} \qquad \text{for} \qquad a, b = 1, 2$$

then (4.1) may be written in the form

$$\mathscr{M}_{1j}(t) = \mathscr{Z}_{1j}(t) + \mathscr{F}_{11} * \mathscr{M}_{1j}(t) \tag{4.3}$$

and

$$\mathscr{M}_{2j}(t) = \mathscr{Z}_{2j}(t) + \mathscr{F}_{21} * \mathscr{M}_{1j}(t) + \mathscr{F}_{22} * \mathscr{M}_{2j}(t), \tag{4.4}$$

$j = 1, 2,..., m$. There are two cases one must consider. If $j \in C_2$ then $\mathscr{Z}_{1j}(t) = \mathscr{O}$ and the solution of (4.3) is $\mathscr{M}_{1j} = \mathscr{O}$. We then see that (4.4) is a system of renewal equations of the irreducible type and Theorem 2.1 may be used to determine the asymptotic behaviour of $\mathscr{M}_{1j}(t)$. However, if $j \in C_1$ we have a genuine reducible case to which we can apply Theorem 3.1. Assuming the hypothesis of Theorem 3.1 is satisfied, an application of the theorem yields the results which are tabulated in Table I. In each case

TABLE I

	Cases	$f_1(t)$	$\mathscr{C}_{1j}$	$f_2(t)$	$\mathscr{C}_{2j}$
1	$\rho_1 = \rho_2 = 1$	1	$\mathscr{B}_1(0) \int \mathscr{L}_{1j}(t)$	t	$\mathscr{B}_2(0)\mathscr{F}_{21}(\infty)\mathscr{B}_1(0) \int \mathscr{L}_{1j}(t)$
2	$\rho_1 = 1, \rho_2 < 1$	1	”	1	$(\mathscr{I}\text{-}\mathscr{F}_{22}(\infty))^{-1}\mathscr{F}_{21}(\infty)\mathscr{B}_1(0) \int \mathscr{L}_{1j}(t)$
3	$\rho_1 < 1, \rho_2 = 1$	$e^{\alpha_1 t}(\alpha_1 < 0)$	”	1	$\mathscr{B}_2(0) \int [\mathscr{L}_{2j}(t) + \mathscr{F}_{21} * \mathscr{M}_{1j}(t)]$
4	$\rho_1 > 1, \alpha_1 > \alpha_2$	$e^{\alpha_1 t}(\alpha_1 > 0)$	$\mathscr{B}_1(\alpha_1) \int e^{-\alpha_1 t} \mathscr{L}_{1j}(t)$	$e^{\alpha_1 t}(\alpha_1 > 0)$	$(\mathscr{I}\text{-}\mathscr{G}_2(\alpha_1))^{-1}\mathscr{G}_{21}(\alpha_1)\mathscr{B}_1(\alpha_1) \int e^{-\alpha_1 t} \mathscr{L}_{1j}(t)$
5	$\rho_1 > 1, \alpha_2 > \alpha_1$	”	”	$e^{\alpha_2 t}(\alpha_2 > 0)$	$\mathscr{B}_2(\alpha_2) \int e^{-\alpha_2 t} [\mathscr{L}_{2j}(t) + \mathscr{F}_{21} * \mathscr{M}_{1j}(t)]$
6	$\rho_1 > 1, \alpha_1 = \alpha_2$	”	”	$te^{\alpha_2 t}(\alpha_2 > 0)$	$\mathscr{B}_2(\alpha_2)\mathscr{G}_{21}(\alpha_2)\mathscr{B}_1(\alpha_1) \int e^{-\alpha_1 t} \mathscr{L}_{1j}(t)$
7	$\rho_1 = 1, \rho_2 > 1$	1	”	$e^{\alpha_2 t}(\alpha_2 > 0)$	Same as 5
8	$\rho_2 > 1, \rho_1 < 1$	$e^{\alpha_1 t}(\alpha_1 < 0)$	”	$e^{\alpha_2 t}(\alpha_2 > 0)$	Same as 5
9	$\rho_1 < 1, \alpha_2 < \alpha_1$	”	”	$e^{\alpha_1 t}(\alpha_1 < 0)$	Same as 4
10	$\rho_2 < 1, \alpha_1 < \alpha_2$	”	”	$e^{\alpha_2 t}(\alpha_2 < 0)$	Same as 5
11	$\rho_1 < 1, \alpha_1 = \alpha_2$	”	”	$te^{\alpha_2 t}(\alpha_2 < 0)$	Same as 6

$[1/f_a(t)]\, \mathscr{M}_{aj}(t) \to \mathscr{C}_{aj}$, $a = 1, 2$, and the number ρ_a is the largest eigenvalue of the matrix $\mathscr{A}_{aa}$, $a = 1, 2$. It is interesting to note that in cases 1, 6, and 11 we get a rate of growth that is not purely exponential.

5. Concluding Remarks

(i) By using Theorem 3.1 or a similar theorem one should be able to obtain convergence-in-mean-square limit theorems for the reducible multidimensional age-dependent branching process somewhat similar to the almost sure type limit theorems obtained by Kesten and Stigum [6] for the reducible (decomposable) multidimensional Galton-Watson process. In a sense the multidimensional age-dependent process seems to be less complex than the multidimensional Galton-Watson process since apparently in the former one need not distinguish between the "positive-regular" and the "periodic" cases.

(ii) The age-dependent process referred to in Section 4 is by no means the only application of the system of equations (1.1). Mode [8] has studied a multidimensional version of the general age-dependent process formulated in [3] to which Theorem 3.1 may be applied.

By considering a branching process in which individuals migrate from one colony to another as in the process studied by Bailey [1] except that there are only $m < \infty$ different colonies, one is led to consider a system of integral equations of the type (1.1). Recently the author has applied Theorem 3.1 to obtain some interesting results for such a process.

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